Journal of Statistical Physics, Vol. 116, Nos. 1/4, August 2004 (© 2004)

Magnetic Lieb–Thirring Inequalities with Optimal Dependence on the Field Strength[1, 2]

László Erdős[3] and Jan Philip Solovej[4]

Received June 27, 2003; accepted October 10, 2003

The Pauli operator describes the energy of a nonrelativistic quantum particle with spin $\frac{1}{2}$ in a magnetic field and an external potential. Bounds on the sum of the negative eigenvalues are called magnetic Lieb–Thirring (MLT) inequalities. The purpose of this paper is twofold. First, we prove a new MLT inequality in a simple way. Second, we give a short summary of our recent proof of a more refined MLT inequality[(8)] and we explain the differences between the two results and methods. The main feature of both estimates, compared to earlier results, is that in the large field regime they grow with the optimal (first) power of the strength of the magnetic field. As a byproduct of the method, we also obtain optimal upper bounds on the pointwise density of zero energy eigenfunctions of the Dirac operator.

KEY WORDS: Kernel of Dirac operator; non-homogeneous magnetic field.

1. INTRODUCTION

1.1. Magnetic Lieb–Thirring Inequalities

Since the seminal paper of Lieb and Thirring,[(14)] Lieb–Thirring inequalities refer to estimates that bound moments of negative eigenvalues of Schrödinger type operators in terms of the external fields. They play a fundamental role in various results concerning localized many-fermion systems. Most notably, the ground state energy of the many-body Hamiltonian

[2] Dedicated to Elliott H. Lieb on his 70th birthday.

[3] School of Mathematics, GeorgiaTech Atlanta, Georgia 30332; e-mail: lerdos@math.gatech.edu; and Mathematisches Institut LMU, Theresienstrasse 39, D-80333 Münich, Germany; e-mail: lerdos@mathematik.uni-muenchen.de

[4] Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen, Denmark; e-mail: solovej@math.ku.dk

in many cases is related to the sum of the negative eigenvalues of an effective one-body Hamiltonian. Among other useful applications, Lieb–Thirring inequalities stand behind the most effective and elegant proofs of stability of matter. They also serve as a powerful apriori estimate for the semiclassical analysis of the many-fermion ground state.

We shall not attempt to give an overview of this vast and beautiful subject, since it is much better to refer the reader to the concise review article of Elliott Lieb.[9] We focus instead on the particular case of *magnetic Lieb–Thirring (MLT) inequalities*. They estimate moments of negative eigenvalues $e_1(H) \leqslant e_2(H) \leqslant \cdots \leqslant 0$ of the Pauli operator

$$H := [\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})]^2 + V \tag{1.1}$$

on $L^2(\mathbf{R}^3, \mathbf{C}^2)$ with a vector potential $\mathbf{A}$, magnetic field $\mathbf{B} := \nabla \times \mathbf{A}$ and external potential V. Here $\boldsymbol{\sigma} \cdot \mathbf{v} = \sigma^1 v_1 + \sigma^2 v_2 + \sigma^3 v_3$, $\mathbf{v} \in \mathbf{R}^3$, and $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices. Unlike in the nonmagnetic case, where the *optimal form* of the estimates is well-established and the remaining main challenge is to find the *optimal constants*, the magnetic case is much less understood. Apart from the case of the constant magnetic field, which has been settled in ref. 12 (in two dimensions[13]), so far there is even no conjecture for a general magnetic Lieb–Thirring inequality that could be considered optimal in all aspects.

Magnetic fields created in laboratories are usually weak and with a fair precision they can be handled perturbatively. Strong magnetic fields, however, occur in certain astrophysical models (e.g., neutron stars, see refs. 11 and 12 and references therein). Even a physically weak magnetic field can become effectively strong in certain problems related to quantum dots.[13]

Both these physical applications and the spirit of universality encompassed in the original Lieb–Thirring inequality have led us to search for magnetic Lieb–Thirring inequalities on the sum of the negative eigenvalues, that are optimal as far as the field strength is concerned in the strong field regime.

The semiclassical formula for the sum of the negative eigenvalues, $\sum_j |e_j(H)|$, for a constant magnetic field behaves linearly in the field strength, $|\mathbf{B}|$ (ref. 12). This fact suggests that $\sum_j |e_j(H)|$ may be bounded by an expression that grows with the first power of $|\mathbf{B}|$ even for nonconstant magnetic fields and away from the semiclassical asymptotic regime. Our goal is to establish such MLT estimates with as few technical assumptions on $\mathbf{B}$ as possible and no technical assumptions on V.

In this paper we present two such estimates. The simpler bound, Theorem 2.1, is proven in this paper. The more involved bound, Theorem 3.2, is outlined in Section 3, but the details of the proof appear elsewhere.[8] We point out that the methods behind these two proofs are very different and they are somewhat complementary, however, we did not succeed in combining the merits of both.

While both bounds are optimal as far as the potential and the strength of the magnetic field are concerned, they require additional technical assumptions on the magnetic field. These are usually formulated in terms of *variation lengthscales*, and practically they are regularity assumptions on $\mathbf{B}$. This means that supremum norms of derivatives of the magnetic field appear in the final Lieb–Thirring inequality.

The difference between the two theorems is that the more involved bound, Theorem 3.2, involves only local supremum norms. Therefore it enjoys an important *locality property*: the estimate is insensitive to the behavior of the magnetic field far away from the support of $[V]_-$, where $[a]_- := -\min\{0, a\}$ denotes the negative part of a. The simpler bound, Theorem 2.1, involves the global C^5-norm of the direction of the magnetic field, $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$. In particular, irregular behaviour of $\mathbf{n}$ far away from the support of $[V]_-$ renders our estimate large despite that it should not substantially influence the negative spectrum. As a compensation, we need less assumptions on the regularity of the field strength $|\mathbf{B}|$ and the proof is much shorter.

Finally we remark that armed with a MLT inequality that scales linearly in the field strength, one may prove the naturally expected semiclassical asymptotics for the sum of the negative eigenvalues of

$$H(h, b) := [\boldsymbol{\sigma}\cdot(-ih\nabla + b\mathbf{A})]^2 + V$$

uniformly in b as $h \to 0$. This requires combining the techniques of refs. 6 and 8 and the details will be published separately.

1.2. Short History of Magnetic Lieb–Thirring Inequalities

For a constant magnetic field, $\mathbf{B} \equiv \text{const}$, the inequality

$$\sum_j |e_j(H)| \leqslant (\text{const}) \left(\int_{\mathbf{R}^3} |\mathbf{B}|\, [V]_-^{3/2} + \int_{\mathbf{R}^3} [V]_-^{5/2} \right) \tag{1.2}$$

proven in ref. 12 is optimal, apart from the constant. It has seemed to be reasonable to conjecture that (1.2) also holds for an arbitrary magnetic field. However, such a naive generalization fails for two reasons.

Firstly, even when $\mathbf{B}$ has constant direction in $\mathbf{R}^3$ (1.2) can be correct only if $|\mathbf{B}(x)|$ is replaced by an effective field strength, $B_{\text{eff}}(x)$, obtained by averaging $|\mathbf{B}|$ locally on the magnetic lengthscale, $|\mathbf{B}|^{-1/2}$.[4]

Secondly, the existence of the celebrated Loss–Yau zero modes[16] contradicts (1.2). Indeed, for certain magnetic fields with nonconstant direction the Dirac operator $\mathscr{D} := \boldsymbol{\sigma}\cdot(-i\nabla+\mathbf{A})$ has a nontrivial L^2-kernel. In this case a small potential perturbation of $\mathscr{D}^2$ shows that $\sum_j |e_j(H)|$ behaves as $\int n(x)[V(x)]_- \, dx$, i.e., it is linear in $[V]_-$. Here $n(x)$ is the density of zero modes, $n(x) = \sum_j |u_j(x)|^2$, where $\{u_j\}$ is an orthonormal basis in Ker $\mathscr{D}$. Thus an extra term linear in $[V]_-$ must be added to (1.2) and $n(x)$ has to be estimated.

The problem of the effective field, observed in ref. 4, was first successfully addressed by Sobolev,[19, 20] and later by Bugliaro *et al.*[1] and Shen.[17] In particular, the L^2-norm of the chosen effective field, $\|B_{\rm eff}\|_2$, is comparable to $\|\mathbf{B}\|_2$ in ref. 1, and the same holds for any L^p-norm in Shen's work. In a very general bound proved in ref. 10 the first term in (1.2) is replaced with $\|\mathbf{B}\|_2^{3/2}\, \|V\|_4$.

In the works[1, 4, 10, 17, 20] the density $n(x)$ is estimated by a function that behaves quantitatively as $|\mathbf{B}(x)|^{3/2}$. In particular, in the strong field regime these estimates are not sufficient to prove semiclassical asymptotics for $H(h, b)$ uniformly in b; they give the asymptotics only up to $b \leqslant$ (const.) h^{-1}.[21]

We remark that the bounds in refs. 1 and 10 have nevertheless been very useful in the proof of magnetic stability of matter. In this case the magnetic energy, $\int |\mathbf{B}|^2$, is also part of the total energy to be minimized, therefore even the second moment of the magnetic field is controlled. We also remark that if the field has a *constant direction*, then no Loss–Yau zero modes exist, $n(x) \equiv 0$. In this case Lieb–Thirring type bounds that grow linearly with $|\mathbf{B}|$ have been proved in refs. 4, 19, and 20.

Therefore the fundamental difficulty is to understand the density of Loss–Yau zero modes. It is amusing to note that it was a substantial endeavour to show that a zero mode may exist at all,[16] and that multiple zero modes may also occur.[7] On the other hand, it seems also quite difficult to give an upper bound on their number in terms of the first power of the field strength (Corollaries 2.2 and 3.3).

Since $n(x)$ scales like $(length)^{-3}$ and $|\mathbf{B}(x)|$ scales like $(length)^{-2}$, a simple dimension counting shows that $n(x)$, therefore $\sum_j |e_j|$, cannot be estimated in general by $|\mathbf{B}(x)|$ or by its smoothed version. However, if an extra lengthscale is introduced, for example certain derivatives of the field are allowed in the estimate, then it is possible to give a bound on the eigenvalue sum that grows slower than $|\mathbf{B}|^{3/2}$ in the large field regime. Before the current paper and our recent work[8] there were only two results in this direction.

The work[3] used a lengthscale on which $\mathbf{B}$ changes. The estimate eventually behaves like $|\mathbf{B}|^{17/12}$ in the field strength. As far as local regularity is concerned, only $\mathbf{B} \in H^1_{\text{loc}}$ is required. However, $n(x)$ is estimated by a quantity that depends *globally* on $\mathbf{B}(x)$ not just in a neighborhood of x. As a consequence, no locality property holds for the MLT inequality in ref. 3.

Our earlier work[5] had a different approach to reduce the power $3/2$ of $|\mathbf{B}|$ in the estimate of $n(x)$. We introduced two *global* lengthscales, L and ℓ respectively, to measure the variation scale of the field strength $|\mathbf{B}|$ and the unit vector $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$ that determines the geometry of the field lines. This required somewhat more regularity on $\mathbf{B}$ than[3] and it also involved the $W^{1,1}$-norm of V. The estimate grew with the $5/4$-th power of the field strength $|\mathbf{B}|$ in the large field regime. For fields with a nearly constant direction, $\ell \gg 1$, the bound was actually better, it behaved like $|\mathbf{B}| + |\mathbf{B}|^{5/4} \ell^{-1/2}$. This indicates that it is only the variation of $\mathbf{n}$ and not that of $|\mathbf{B}|$ that is responsible for the higher $|\mathbf{B}|$-power. The MLT estimate in ref. 5 did not enjoy the locality property either.

Due to the improvement in the $|\mathbf{B}|$-power from $3/2$ to $5/4$ in the MLT estimate we could also prove the semiclassical eigenvalue asymptotics for $H(h, b)$ in the regime $b \ll h^{-3}$ for potentials in $W^{1,1}$.[6] This bound turned out to be sufficient to show that the Magnetic Thomas-Fermi theory exactly reproduces the ground state energy of a large atom with nuclear charge Z in the semiclassical regime, i.e., where $b \ll Z^3$, $Z \to \infty$.[6] The condition $b \ll Z^3$ is optimal as far as the semiclassical theory is applicable.

Despite the successful application of the bound in ref. 3 to the stability of matter with quantized electromagnetic field with an ultraviolet cutoff,[2] and despite that the MLT inequality given in ref. 5 fully covered the semiclassical regime of large atoms, it is still important to establish a uniform Lieb–Thirring type bound with the correct power in the magnetic field and with no unnecessary assumptions on V. Such bound will likely be the key to generalize the analysis of the super-strong field regime of ref. 11 to non-homogeneous magnetic fields.

2. LIEB-THIRRING INEQUALITY WITHOUT LOCALITY PROPERTY

We consider the three dimensional Pauli operator, $H = \mathcal{D}^2 + V$, with a differentiable magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The operator

$$\mathcal{D} := \boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A}) \tag{2.1}$$

is called the Dirac operator with vector potential $\mathbf{A}$ and magnetic field $\mathbf{B}$. We make two global assumptions:

Assumption 1. $\mathbf{B}(x) \neq 0$ for all $x \in \mathbf{R}^3$, i.e., the unit vectorfield $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$ is well defined.

Assumption 2. The vectorfield $\mathbf{n}$ satisfies the following global regularity condition

$$L_{\mathbf{n}}^{-1} := \sum_{\gamma=1}^{5} \|\nabla^\gamma \mathbf{n}\|_\infty^{1/\gamma} < \infty, \tag{2.2}$$

where $L_{\mathbf{n}}$ is called the *global variation lengthscale* of $\mathbf{n}$.

For any $L > 0$, $x \in \mathbf{R}^3$ we also define

$$B_L^*(x) := \sup\{|\mathbf{B}(y)| : |y-x| \leqslant L\} + L \cdot \sup\{|\nabla \mathbf{B}(y)| : |y-x| \leqslant L\}.$$

Theorem 2.1 (Lieb–Thirring Inequality without a Locality Property). For any $0 < L \leqslant L_{\mathbf{n}}$, the sum of the negative eigenvalues, $e_1(H) \leqslant e_2(H) \leqslant \cdots \leqslant 0$, of H satisfies

$$\sum_j |e_j(H)| \leqslant c \left(L^{-1} \int (B_L^* + L^{-2})[V]_- + \int B_L^* [V]_-^{3/2} + \int [V]_-^{5/2} \right) \tag{2.3}$$

with a universal constant c.

Let $\Pi_{\mathscr{D}^2 \leqslant \lambda}$ denote the spectral projection onto energy levels below λ in the spectrum of $\mathscr{D}^2$. Along the proof of Theorem 2.1 we obtain the following bound on the density of the low lying states of $\mathscr{D}^2$:

Corollary 2.2 (Density of Low Lying States). For any $0 < L \leqslant L_{\mathbf{n}}$ and $x \in \mathbf{R}^3$

$$\Pi_{\mathscr{D}^2 \leqslant \lambda}(x, x) \leqslant c \lambda_*^{1/2} \max\{B_L^*(x), \lambda_*\}, \qquad \text{with} \quad \lambda_* := \max\{\lambda, L^{-2}\}, \tag{2.4}$$

in particular the local density of zero modes of $\mathscr{D}^2$ grows with at most the first power of the field strength.

Convention. We use the letter c to denote various positive universal constants whose exact value may change from line to line.

3. LIEB–THIRRING INEQUALITY WITH A LOCALITY PROPERTY

In this Section we give an outline of a more refined magnetic Lieb–Thirring inequality. The detailed proof appears elsewhere.[8]

We assume that $\mathbf{B} \in C^4(\mathbf{R}^3, \mathbf{R}^3)$, and define three basic lengthscales of $\mathbf{B}$. The Pauli operator will be localized on these lengthscales. Let $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$ be the unit vectorfield in the direction of the magnetic field at all points where $\mathbf{B}$ does not vanish. For any $L > 0$ and $x \in \mathbf{R}^3$ we define

$$B_L(x) := \sup\{|\mathbf{B}(y)|: |x-y| \leqslant L\}, \tag{3.1}$$

and

$$b_L(x) := \inf\{|\mathbf{B}(y)|: |x-y| \leqslant L\} \tag{3.2}$$

to be the supremum and the infimum of the magnetic field strength on the ball of radius L about x.

Definition 3.1 (Lengthscales of a Magnetic Field). The **magnetic lengthscale** of $\mathbf{B}$ is defined as

$$L_m(x) := \sup\{L > 0 : B_L(x) \leqslant L^{-2}\}.$$

The **variation lengthscale** of $\mathbf{B}$ at x is given by

$$L_v(x) := \sup\{L \geqslant 0 : L^\gamma \sup\{|\nabla^\gamma \mathbf{B}(y)| : |x-y| \leqslant L\} \leqslant b_L(x), \gamma = 1, 2, 3, 4\}.$$

Finally we set

$$L_c(x) := \max\{L_m(x), L_v(x)\}. \tag{3.3}$$

A magnetic field $\mathbf{B}: \mathbf{R}^3 \to \mathbf{R}^3$ determines three local lengthscales. The magnetic lengthscale, L_m, is comparable with $|\mathbf{B}|^{-1/2}$. The lengthscale L_s determines the scale on which the strength of the field varies, i.e., it is the variation scale of $\log |\mathbf{B}|$. The field line structure, determined by $\mathbf{n}$, varies on the scale of L_n. The variation lengthscale L_v is the smaller of these last two scales, i.e., it is the scale of variation of the vectorfield $\mathbf{B}$.

For weak magnetic fields the magnetic effects can be neglected in our final eigenvalue estimate, so the variational lengthscale becomes irrelevant. This idea is reflected in the definition of L_c; we will not need to localize on scales shorter than the magnetic scale L_m.

The following theorem is the main result in ref. 8.

Theorem 3.2 (Lieb–Thirring Inequality with a Locality Property). We assume that the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ is in $C^4(\mathbf{R}^3,\mathbf{R}^3)$. Then the sum of the negative eigenvalues of $H=[\boldsymbol{\sigma}\cdot(-i\nabla+\mathbf{A})]^2+V$ satisfies

$$\sum_j |e_j| \leqslant c\int [V]_-^{5/2}+c\int |\mathbf{B}|\,[V]_-^{3/2}+c\int (|\mathbf{B}|+L_c^{-2})\,L_c^{-1}[V]_-. \tag{3.4}$$

Corollary 3.3 (Density of Zero Modes with a Locality Property). Given a magnetic field $\mathbf{B}\in C^4(\mathbf{R}^3,\mathbf{R}^3)$, the density of zero modes of the free Dirac operator $\mathscr{D}$ with magnetic field $\mathbf{B}$ satisfies

$$n(x):=\sum_j |u_j(x)|^2\leqslant c(|\mathbf{B}(x)|+L_c^{-2}(x))\,L_c^{-1}(x), \tag{3.5}$$

where $\{u_j\}$ is an orthonormal basis in the kernel of $\mathscr{D}=\boldsymbol{\sigma}\cdot(-i\nabla+\mathbf{A})$.

Remarks. (i) The bound (3.5) is optimal as far as the strength of the field $|\mathbf{B}|$ is concerned. This fact follows from the construction of Dirac operators with kernels of high multiplicity following the method of ref. 7.

(ii) Notice that the Lieb–Thirring inequality of ref. 12 for a *constant* field is recovered in Theorem 3.2. However, the uniform Lieb–Thirring bound for a *constant direction* field,[5, 20] does not directly follow from our main theorem as it is stated. Firstly, (3.4) contains a term linear in V that is unnecessary for a constant direction field. Secondly, we assume high regularity on $\mathbf{B}$. This regularity is needed only to construct the appropriate curvilinear cylindrical localization, which is unnecessary for a field with constant direction.

(iii) It can be shown that $L_c(x)$ enjoys a locality property; for any $\delta>0$ the inverse lengthscale, $L_c^{-1}(x)$, is bounded by a function depending on δ and on the magnetic field in a δ-neighborhood of x. In particular, $L_c(x)^{-1}\leqslant c\delta^{-1}$ if $\mathbf{B}$ vanishes in a δ-neighborhood of x.

Before we turn to the proof of our new result, Theorem 2.1, we explain the key difference between the two proofs that is somewhat hidden behind technicalities.

The linear power of $|\mathbf{B}|$ in the estimate reflects the basic fact that the space with a magnetic field cannot be considered isotropic: the quantum motion parallel with the magnetic field behaves differently than the transversal one. The magnetic field affects only the transversal motion; this is why the eigenvalue sum scales in the same (linear) power of $|\mathbf{B}|$ both in $d=2$ and 3 dimensions. The motion in the third direction affects only the powers of the potential, as it is customary in the nonmagnetic Lieb–Thirring inequalities.

All MLT estimates that yield $|\mathbf{B}|^{3/2}$ behaviour neglect this geometric fact by simply comparing the magnetic problem with a nonmagnetic one, usually via a diamagnetic inequality that loses the anisotropic feature of the problem. The typical estimate is of the form

$$\mathscr{D}^2 \geqslant b^{-1}\mathscr{D}^2 = b^{-1}(\mathbf{D}^2 + \boldsymbol{\sigma}\cdot\mathbf{B}) \tag{3.6}$$

where $\mathbf{D} := -i\nabla + \mathbf{A}$ is the spinless magnetic momentum and $b := \|\mathbf{B}\| \gg 1$ denote some (local) norm of $\mathbf{B}$. The Pauli kinetic energy is scaled down so that the dangereous $\boldsymbol{\sigma}\cdot\mathbf{B}$ becomes bounded uniformly in b. The magnetic Laplacian $\mathbf{D}^2$ then can be controlled by the nonmagnetic Laplacian, $-\Delta$, but the factor b^{-1} now affects all three coordinates, yielding a scaling of $b^{3/2}$. The key to the improvement is to separate the motion in parallel and in the transversal directions and use a crude estimate similar to (3.6) only in the two-dimensional transversal kinetic energy.

Since the direction of the magnetic field $\mathbf{n}$ varies, it is in general impossible to introduce global orthogonal coordinates parallel and transversal to $\mathbf{n}$. The more conventional approach is the one presented in ref. 8: we introduce appropriate coordinates locally and approximate the magnetic field by another one that is constant in these coordinates. For constant magnetic field the separation is then trivial. The main technical challenge is to keep all approximation and localization errors bounded uniformly in the field strength b. This requires many steps: most importantly we have to change the metric by a conformal factor to make the field strength constant along the field lines; we have to localize the problem onto the physically correct domains, which are elongated curvilinear cylinders of width $b^{-1/2}$ along the field lines and we have to construct appropriate coordinate systems along each field line. The construction steps require tools from the spinor geometry behind the Dirac operators. The localization onto narrow cylinders is done via the magnetic localization formula from.[(6)] All these steps require fairly high regularity on the magnetic field.

The more direct approach is the one behind Theorem 2.1. This method bypasses all the complications with the constant field approximation, the cylindrical construction and the change of metric. It uses the simple observation that the energy in the parallel direction can be extracted from $\mathscr{D}^2$ by a simple energy argument (see (6.18) in Lemma 6.2):

$$\mathscr{D}^2 \geqslant cD_{\mathbf{n}}^2 - (\text{error}), \tag{3.7}$$

where $D_{\mathbf{n}} := -i\partial_{\mathbf{n}} - \frac{i}{2}\,\mathrm{div}\,\mathbf{n}$, and the error terms are of lower order in momentum. It turns out to be necessary to consider higher powers of the resolvent $(\mathscr{D}^2+\lambda)^{-1}$ to control the ultraviolet regime (here $\lambda>0$ is a constant controlled by the geometry of the field lines), therefore it is necessary to estimate higher powers of $\mathscr{D}^2$ as well, the key observation is that after having extracted $D_{\mathbf{n}}^2$, higher powers of the crude diamagnetic estimate (3.6) are sufficient. Although (3.6) cannot be squared directly, after several commutators it is possible to show that (see (6.19))

$$\mathscr{D}^{2k}\geqslant[\delta\mathbf{D}^2]^k-(\text{error}), \tag{3.8}$$

where $\delta:=\min\{\lambda b^{-1}, 1\}$.

Therefore the basic estimate for the density of the low lying states is

$$\Pi_{\mathscr{D}^2\leqslant\lambda}\leqslant\frac{8\lambda^3}{(\mathscr{D}^2+\lambda)^3}\leqslant\frac{c\lambda^3}{\mathscr{D}^6+\lambda^2D_{\mathbf{n}}^2+\lambda^3}\leqslant\frac{c\lambda^3}{(\delta\mathbf{D}^2)^3+\lambda^2D_{\mathbf{n}}^2+\lambda^3}.$$

Then we can essentially use the diamagnetic inequality (although first we have to ensure that the two operators in the denominator commute, see Lemma 6.1) and the diagonal term is estimated as

$$\Pi_{\mathscr{D}^2\leqslant\lambda}(x,x)\leqslant\frac{c\lambda^3}{\delta^3(-\Delta)^3+\lambda^2\partial_{\mathbf{n}}^*\partial_{\mathbf{n}}+\lambda^3}(x,x).$$

It is then easy to change this quantity into Euclidean coordinates and compute

$$\int_{\mathbf{R}^3}\frac{\lambda^3}{\delta^3(\mathbf{p}^2)^3+\lambda^2p_3^2+\lambda^3}\,\mathrm{d}\mathbf{p}\leqslant c\lambda^{1/2}(b+\lambda)$$

which scales linearly with b.

Apart from several technicalities, there is an apparently innocent but fundamental complication in both approaches. Both proofs uses the concept of the local magnetic field strength, b, that is given by some local norm of $\mathbf{B}$. Therefore at the beginning one needs to localize the original operator $\mathscr{D}^2$ onto domains of size independent of b to establish this apriori localization scale. We choose domains of size $L_{\mathbf{n}}$ where the field lines do not vary substantially.

Localizing higher powers of $\mathscr{D}^2$ even onto the scale $L_{\mathbf{n}}$ is difficult because it does not follow simply from energy considerations. If the localization length $L_{\mathbf{n}}$ is a uniform constant then it can be done using the spectral theorem (see (4.2)). This is the method in Theorem 2.1, but it forces one to introduce the same localization error everywhere in space, even in regimes, where the field is very regular.

If one wants to include a variable $L_{\mathbf{n}}$, as in Theorem 3.2, to ensure the locality property, then one needs a more powerful apriori localization. Typically it is not hard to localize resolvents of second order elliptic operators onto cubes of size ℓ at the expense of an error ℓ^{-2}. However localizing the square (or higher powers) of the resolvent requires off-diagonal estimates on the resolvent kernel (see Proposition 7.1 in ref. 8). While these are typically easily available for scalar elliptic operators without spin, we *do not know any apriori off-diagonal control on the resolvent of* $\mathscr{D}^2$. If the original Pauli operator is estimated by a constant field Pauli operator, then *aposteriori* we can extract off-diagonal estimates, and this idea is used, though very implicitly, in ref. 8. But without comparison with the constant field problem, we do not have off-diagonal control. This is the main reason why we are unable to extend the elegant and short method of Theorem 2.1 to give any locality properties.

4. ORGANIZATION OF THE PROOF OF THEOREM

We can assume that the potential in (1.1) is nonpositive, so for simplicity we can consider $H = \mathscr{D}^2 - V$ with $V \geqslant 0$. We start with the Birman–Schwinger principle

$$|\mathrm{Tr}(\mathscr{D}^2 - V)_-| = 2\int_0^\infty n\left(|V-E|_-^{1/2}\,\frac{1}{\mathscr{D}^2+E}\,|V-E|_-^{1/2},\,1\right)\mathrm{d}E, \qquad (4.1)$$

where $n(X; c)$ denotes the number of eigenvalues of the positive self-adjoint operator X above level c.

We estimate $\mathscr{D}^2$ from below by the magnetic Laplacian, then use the diamagnetic inequality to estimate its resolvent kernel by that of the free Laplacian. These estimates involve error terms that cannot be controlled by E in the resolvent $(\mathscr{D}^2+E)^{-1}$ when E is small. As a first step, we insert a positive constant P in the resolvent using the inequality

$$\frac{1}{\mathscr{D}^2+E} \leqslant \frac{2}{\mathscr{D}^2+P+E} + \frac{8}{E}\,\frac{P^3}{(\mathscr{D}^2+P)^3} \qquad (4.2)$$

that follows from the spectral theorem and the corresponding arithmetic inequality. P is chosen as

$$P := \varepsilon^{-4} L^{-2},$$

where ε denotes a sufficiently small universal constant.

From (4.2) and the general bound $n(X_1+X_2; c_1+c_2) \leqslant n(X_1, c_1) + n(X_2, c_2)$, we obtain

$$|\mathrm{Tr}(\mathscr{D}^2 - V)_-| \leqslant c[(I)+(II)] \tag{4.3}$$

with

$$(I) := \int_0^\infty n\left(|V-E|_-^{1/2} \frac{1}{\mathscr{D}^2+P+E} |V-E|_-^{1/2}, \frac{1}{4}\right) \mathrm{d}E; \tag{4.4}$$

$$(II) := \int_0^\infty n\left(|V-E|_-^{1/2} \frac{P^3}{(\mathscr{D}^2+P)^3} |V-E|_-^{1/2}, \frac{1}{16} E\right) \mathrm{d}E. \tag{4.5}$$

These two terms will be called *positive energy regime* and *zero mode regime*, respectively (see ref. 8). The proof of Theorem 2.1 then follows from the two estimates below:

Proposition 4.1. For sufficiently small ε

$$(I) \leqslant c \int B_L^*(x)\, V^{3/2}(x)\, \mathrm{d}x + c \int V^{5/2} \tag{4.6}$$

and

$$(II) \leqslant c P^{1/2} \int \max\{B_L^*(x), P\}\, V(x)\, \mathrm{d}x. \tag{4.7}$$

To obtain these estimates, we will have to bound the resolvent of $\mathscr{D}^2$ and the diagonal kernel of the cube of the resolvent of $\mathscr{D}^2$ (see (7.2)). In both cases we localize the operator $\mathscr{D}^2$ onto cubes of size of order L^{-1}.

In Section 5 we define localization functions and new Dirac operators which coincide with $\mathscr{D}$ locally but have a uniformly bounded magnetic field everywhere. In Section 6 we prove various estimates on a Dirac operator with a bounded magnetic field and we compare it with an auxiliary Laplacian-type elliptic operator that is diagonal in the spin space. Finally, in Sections 7 and 8 we complete the estimates (4.7) and (4.6), respectively.

5. LOCALIZATION

Consider the cubic lattice $\Lambda = \varepsilon L \mathbf{Z}^3$. For each $k \in \Lambda$, let $\Omega_k \subset \Omega_k' \subset \Omega_k^* \subset \Omega_k^\#$ be the balls with center k and with radius εL, $2\varepsilon L$, $3\varepsilon L$, and $4\varepsilon L$, respectively. Let $\{\theta_k\}_{k\in\Lambda}$ be a partition of unity such that $0\leqslant\theta_k\leqslant 1$,

$$\sum_{k\in\Lambda}\theta_k^2\equiv 1,\quad \operatorname{supp}\theta_k\subset\Omega_k,\quad \|\nabla^\gamma\theta_k\|_\infty\leqslant c(\varepsilon L)^{-\gamma},\quad 1\leqslant\gamma\leqslant 5. \tag{5.1}$$

We also choose further localization functions η_k, ω_k and χ_k with $\theta_k\leqslant\eta_k\leqslant\omega_k\leqslant\chi_k\leqslant 1$, $k\in\Lambda$, such that

$$\operatorname{supp}\eta_k\subset\Omega'_k,\quad \eta_k\equiv 1\quad\text{on }\Omega_k,\quad \|\nabla^\gamma\eta_k\|_\infty\leqslant c(\varepsilon L)^{-\gamma},\quad 1\leqslant\gamma\leqslant 5; \tag{5.2}$$

$$\operatorname{supp}\omega_k\subset\Omega^*_k,\quad \omega_k\equiv 1\quad\text{on }\Omega'_k,\quad \|\nabla^\gamma\omega_k\|_\infty\leqslant c(\varepsilon L)^{-\gamma},\quad 1\leqslant\gamma\leqslant 5; \tag{5.3}$$

$$\operatorname{supp}\chi_k\subset\Omega^\#_k,\quad \chi_k\equiv 1\quad\text{on }\Omega^*_k,\quad \|\nabla^\gamma\chi_k\|_\infty\leqslant c(\varepsilon L)^{-\gamma},\quad 1\leqslant\gamma\leqslant 5. \tag{5.4}$$

We note that

$$\sum_{k\in\Lambda}\eta_k\leqslant C,\qquad \sum_{k\in\Lambda}\omega_k\leqslant C,\qquad \sum_{k\in\Lambda}\chi_k\leqslant C \tag{5.5}$$

for some universal constant C by the finite overlap property of the balls.

Let $b_k:=\sup_{\Omega^\#_k}(|\mathbf{B}|+L\,|\nabla\mathbf{B}|)$, then we have

$$|\mathbf{B}|+L\,|\nabla\mathbf{B}|\leqslant b_k\leqslant B^*_L\quad\text{on }\operatorname{supp}(\chi_k), \tag{5.6}$$

if ε is sufficiently small.

The following lemma defines a new Dirac operator for each cube Ω_k that coincides with $\mathscr{D}$ on Ω^*_k but has a uniformly bounded magnetic field. The proof is given in Section A.1.

Lemma 5.1. For each $k\in\Lambda$ there exists a nowhere vanishing magnetic field $\mathbf{B}_k$ such that $\mathbf{B}_k\equiv\mathbf{B}$ on Ω^*_k, $\mathbf{B}_k\equiv(\text{const})$ on $\mathbf{R}^3\setminus\Omega^\#_k$,

$$\|\mathbf{B}_k\|_\infty+L\,\|\nabla\mathbf{B}_k\|_\infty\leqslant cb_k, \tag{5.7}$$

and $\mathbf{n}_k:=\mathbf{B}_k/|\mathbf{B}_k|$ satisfies the bound

$$\sum_{\gamma=1}^{5}(\varepsilon L)^\gamma\,\|\nabla^\gamma\mathbf{n}_k\|_\infty\leqslant c\varepsilon. \tag{5.8}$$

Moreover, there exists a vector potential $\mathbf{A}_k$, $\nabla\times\mathbf{A}_k=\mathbf{B}_k$ such that $\mathbf{A}_k\equiv\mathbf{A}$ on Ω^*_k. The Dirac operator $\mathscr{D}_k:=\boldsymbol{\sigma}\cdot(-i\nabla+\mathbf{A}_k)$ with magnetic field $\mathbf{B}_k$ therefore satisfies

$$\mathscr{D}_k\eta_k=\mathscr{D}\eta_k. \tag{5.9}$$

The localization is performed by the following "Pull-up" formula whose proof was given in ref. 1 and was also recalled in Proposition 6.1. in ref. 8.

Proposition 5.2 (Pull-Up Formula). Let I be a countable index set and let g_i, $i \in I$, be a family of nonnegative functions on $\mathbf{R}^3$ such that $\sum_{i\in I} g_i^2(x) < \infty$ for every $x \in \mathbf{R}^3$. Let A_i, $i \in I$, be a family of positive invertible self-adjoint operators on $L^2(\mathbf{R}^3; \mathbf{C}^2)$. Then

$$\Big(\sum_{i\in I} g_i^2\Big)\frac{1}{\sum_{i\in I} g_i A_i g_i}\Big(\sum_{i\in I} g_i^2\Big) \leqslant \sum_{i\in I} g_i \frac{1}{A_i} g_i. \tag{5.10}$$

The resolvent $(\mathscr{D}^2+P+E)^{-1}$ can be easily localized using the partition of unity $\{\theta_k\}$.

$$\mathscr{D}^2 = \sum_k \mathscr{D}\theta_k^2\mathscr{D} \geqslant \tfrac{1}{2}\sum_k \theta_k\mathscr{D}^2\theta_k - \sum_k |\nabla\theta_k|^2 \geqslant \tfrac{1}{2}\sum_k \theta_k\mathscr{D}_k^2\theta_k - c\varepsilon^{-2}L^{-2}$$

using (5.9), the finite overlap property of the supports of θ_k's, and the estimate on their derivatives (5.1). For sufficiently small ε the error can be absorbed into P, so

$$\mathscr{D}^2+P+E \geqslant \tfrac{1}{2}\sum_k \theta_k(\mathscr{D}_k^2+P+E)\,\theta_k.$$

Using the "Pull-up" formula (5.10), we obtain

$$\frac{1}{\mathscr{D}^2+P+E} \leqslant \frac{2}{\sum_k \theta_k(\mathscr{D}_k^2+P+E)\,\theta_k} \leqslant 2\sum_k \theta_k \frac{1}{\mathscr{D}_k^2+P+E}\theta_k. \tag{5.11}$$

Similar localization does not hold for higher powers of the resolvent as it was remarked in Section 7 of ref. 8. However, we can localize the second term in (4.2) estimating it by an auxiliary elliptic operator that is independent of the spin coordinates. The construction is given in Section 6.

6. ANALYSIS OF BOUNDED MAGNETIC FIELDS

Throughout this section we assume that a vector potential $\mathbf{A}$ is given so that for some positive constants L, ε, and b, the nowhere vanishing magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ of the Dirac operator $\mathscr{D} = \boldsymbol{\sigma}\cdot(-i\nabla+\mathbf{A})$ satisfies the following conditions:

$$\mathbf{B} \equiv \text{const} \qquad \text{outside of } \Omega^{\#}, \tag{6.1}$$

where $\Omega^{\#}$ is a ball of size $O(\varepsilon L)$;

$$0 < |\mathbf{B}| + L\, |\nabla \mathbf{B}| \leqslant b \tag{6.2}$$

and

$$\sum_{\gamma=1}^{5} (\varepsilon L)^{\gamma}\, \|\nabla^{\gamma} \mathbf{n}\|_{\infty} \leqslant c\varepsilon, \tag{6.3}$$

where $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$. The plane $\mathscr{P}$ through the center of $\Omega^{\#}$ and orthogonal to the magnetic field at the center is called the supporting plane.

The results of this section will be applied to the Dirac operators $\mathscr{D}_k$ with a magnetic field $\mathbf{B}_k$ constructed in Lemma 5.1, but we do not carry the index k inside this section.

We choose a global, positively oriented orthonormal basis $\{e_1, e_2, e_3\}$ in $\mathbf{R}^3$ whose third vector is $e_3 := \mathbf{n}$, and such that all Christoffel symbols $\Gamma^m_{kj} = (\nabla_{e_j} e_k, e_m)$ vanish outside the ball twice bigger than $\Omega^{\#}$, and they satisfy the estimate

$$\sum_{\gamma=0}^{4} (\varepsilon L)^{\gamma}\, \|\nabla^{\gamma} \Gamma^m_{kj}\|_{\infty} \leqslant cL^{-1}. \tag{6.4}$$

That such and othonormal basis exists follows easily from (6.3), e.g., by a Gram-Schmidt procedure starting from $\mathbf{n}, \widetilde{e}_1, \widetilde{e}_2$, where $\widetilde{e}_1, \widetilde{e}_2$ are fixed vectors. We will work on the trivial spinorbundle over $\mathbf{R}^3$, i.e., in the $L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$ space. We briefly recall the appropriate formalism from Section 9 of ref. 8.

For any 1-form α we define the spinor connection

$$\nabla^{\alpha}_X := \partial_X + i\alpha(X) + \frac{i}{2}\, \boldsymbol{\sigma} \cdot \boldsymbol{\omega}(X),$$

where

$$\boldsymbol{\omega}(X) := ((\nabla_X e_3, e_2), (\nabla_X e_1, e_3), (\nabla_X e_2, e_1)),$$

and let

$$\Pi^{\alpha}_j := -i\nabla^{\alpha}_{e_j}, \qquad j = 1, 2, 3.$$

For simplicity we will use $\nabla_j := \nabla_{e_j}$ and $\partial_j = \partial_{e_j}$ and we usually omit the superscript α from the notation. We recall that the operator

$$\hat{D}_X := -i\partial_X - \frac{i}{2} \operatorname{div} X \tag{6.5}$$

is self-adjoint on $L^2(\mathbf{R}^3)$ for any C^1 bounded vectorfield X, therefore the operator

$$D_X := -i\partial_X - \frac{i}{2}\operatorname{div} X + \alpha(X) \tag{6.6}$$

is self-adjoint. The hat refers to the operators without magnetic field, $\alpha \equiv 0$.

These operators are originally defined on $L^2(\mathbf{R}^3)$, but with a little abuse of notations they can also be viewed as acting on $L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$ as $D_X \otimes I_2$, where I_2 is the identity operator on the spin space. We will not distinguish between these two operators in the notations. We set $D_j := D_{e_j}$ for simplicity, then

$$\Pi_j = D_j + \frac{i}{2}\operatorname{div} e_j + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}(e_j). \tag{6.7}$$

We compute the commutators of the D_j's:

$$[D_j, D_k] = (-i)(D_{[e_j, e_k]} + \beta_{jk}) \tag{6.8}$$

where $\beta := \mathrm{d}\alpha$ and $\beta_{jk} := \beta(e_j, e_k)$. In the computation we used that the divergence part vanishes since

$$X \operatorname{div} Y - Y \operatorname{div} X - \operatorname{div}[X, Y] = 0 \tag{6.9}$$

for any vectorfields. The relation (6.9) is easy to check on coordinate vectorfields and it is tensorial.

Let $Q^m_{jk} := \Gamma^m_{kj} - \Gamma^m_{jk}$, then $D_{[e_j, e_k]} = \frac{1}{2}(Q^m_{jk} D_m + D_m Q^m_{jk})$, where we used the summation convention for repeated indices. From (6.8)

$$[D_j, D_k] = -\frac{i}{2}(Q^m_{jk} D_m + D_m Q^m_{jk}) - i\beta_{jk} = (-i)\left(Q^m_{jk} D_m - \frac{i}{2}(\partial_m Q^m_{jk}) + \beta_{jk}\right). \tag{6.10}$$

We recall from Section 9 of ref. 8 that ∇^α_X is a spinor connection with a magnetic 2-form $\beta := \mathrm{d}\alpha$. If we choose the vector potential $\alpha(X) := \mathbf{A}\cdot X$, then $\star\beta$ is the 1-form corresponding to the vectorfield $\mathbf{B}$, where $\star$ denotes the Hodge dual with respect to the standard Euclidean metric. With these notations the Dirac operator $\mathscr{D} = \boldsymbol{\sigma}\cdot(-i\nabla + \mathbf{A})$ in Euclidean coordinates is $SU(2)$-gauge equivalent to

$$\tilde{\mathscr{D}} := \boldsymbol{\sigma}\cdot\boldsymbol{\Pi}, \tag{6.11}$$

i.e.,

$$\mathcal{D} = U\tilde{D}U^* \tag{6.12}$$

for some $U: \mathbf{R}^3 \mapsto SU(2)$. Tilde always refers to Dirac operators written in a basis where the third basis element is parallel with the magnetic field. In the chosen basis the magnetic 2-form satisfies $\beta(e_1, e_2) = B$, where $B := |\mathbf{B}|$ is the field strength, and all other $\beta(e_j, e_k) = 0$.

The Lichnerowicz formula is given by

$$\tilde{\mathcal{D}}^2 = \mathbf{\Pi}^* \cdot \mathbf{\Pi} + \tfrac{1}{4} R_0 + \sigma(\star\beta), \tag{6.13}$$

where R_0 is the scalar curvature (which vanishes in our case) and for any one-form $\lambda = \sum_j \lambda_j e^j$ we define $\sigma(\lambda) = \sum_j \sigma^j \lambda_j$.

Let Z be a symmetric positive definite 3 by 3 matrix valued function on $\mathbf{R}^3$ that satisfies

$$\|Z - I\| + \sum_{\gamma=1}^{3} (\varepsilon L)^\gamma \|\nabla^\gamma Z\| \leqslant c\varepsilon, \tag{6.14}$$

and let h be a function with

$$\sum_{\gamma=0}^{2} (\varepsilon L)^\gamma \|\nabla^\gamma h\|_\infty \leqslant c\varepsilon^{-1} L^{-2}. \tag{6.15}$$

We define

$$T := T_{Z,h} := \mathbf{D} \cdot Z\mathbf{D} + h, \qquad \hat{T} := \hat{T}_{Z,h} := \hat{\mathbf{D}} \cdot Z\hat{\mathbf{D}} + h \tag{6.16}$$

on $L^2(\mathbf{R}^3)$, but we will also denote simply by $T, \hat{T}$ the operators $T \otimes I_2$ and $\hat{T} \otimes I_2$ acting on $L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$. We will choose Z and h according to the following lemma whose proof is given in Section A.2.

Lemma 6.1. Let the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ satisfy (6.1), (6.2), and (6.3). Then for sufficiently small ε there exists a 3 by 3 symmetric matrix valued function Z and a real function h satisfying (6.14) and (6.15), such that the commutator of $T = \mathbf{D} \cdot Z\mathbf{D} + h$ and $D_{\mathbf{n}}$ vanishes,

$$[T, D_{\mathbf{n}}] = 0. \tag{6.17}$$

Armed with these notations, we state the following lower bounds on the Pauli operator. The proof is given in Section A.3.

Lemma 6.2. Let the magnetic field $\mathbf{B}$ of the Dirac operator $\tilde{\mathcal{D}} = \boldsymbol{\sigma} \cdot \mathbf{\Pi}$ satisfy (6.1), (6.2), and (6.3). Let $0 \leqslant \theta \leqslant \eta \leqslant \omega \leqslant \chi \leqslant 1$ be real functions on $\mathbf{R}^3$ with $|\nabla^\gamma\theta|, |\nabla^\gamma\eta|, |\nabla^\gamma\omega|, |\nabla^\gamma\chi| \leqslant c(\varepsilon L)^{-\gamma}$, $1\leqslant\gamma\leqslant 5$, and $\eta\equiv 1$ on $\mathrm{supp}(\theta)$, $\omega\equiv 1$ on $\mathrm{supp}(\eta)$ and $\chi\equiv 1$ on $\mathrm{supp}(\omega)$. Let $P:=\varepsilon^{-4}L^{-2}$, $\lambda\geqslant P$ and $\delta:=\min\{\lambda b^{-1}, 1\}$. Then for sufficiently small ε there exists a positive universal constant c such that

$$\tilde{\mathscr{D}}^2 \geqslant cD_{\mathbf{n}}^2 - \varepsilon^2 P \tag{6.18}$$

and

$$\tilde{\mathscr{D}}^3\theta^2\tilde{\mathscr{D}}^3 + \lambda\tilde{\mathscr{D}}^2\eta^2\tilde{\mathscr{D}}^2 + \lambda^2\tilde{\mathscr{D}}\omega^2\tilde{\mathscr{D}} + \lambda^3\chi^2 \geqslant c\theta(\delta T)^3\,\theta, \tag{6.19}$$

where $T = T_{Z,h}$ is defined in (6.16) with the bounds (6.14), (6.15).

Finally, we estimate the diagonal of the resolvent kernel of the sum of the lower bounds obtained in Lemma 6.2. The proof is given in A.4.

Lemma 6.3. Let the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ satisfy (6.1), (6.2), and (6.3), and let $\lambda\geqslant\varepsilon^{-4}L^{-2}$, $\delta:=\min\{\lambda b^{-1}, 1\}$. Then for sufficiently small ε

$$\frac{1}{(\delta T)^3 + \lambda^2 D_{\mathbf{n}}^2 + \lambda^3}(x, x) \leqslant c\lambda^{1/2}\max\{b, \lambda\}. \tag{6.20}$$

7. PROOF OF THE ZERO MODE REGIME

In this section we prove a Lemma that immediately implies (4.7). The main tool is a localization of the third power of the resolvent. For later purposes, we will need the following, somewhat more general estimate:

Lemma 7.1. Assume that the magnetic field $\mathbf{B}$ of a Dirac operator $\mathscr{D} = \boldsymbol{\sigma}\cdot(-i\nabla + \mathbf{A})$ is nowhere vanishing and that it has a finite, nonzero global variation lengthscale $L_{\mathbf{n}}$ (2.2). Let $L\leqslant L_{\mathbf{n}}$, then for sufficiently small ε and for any $\lambda\geqslant P = \varepsilon^{-4}L^{-2}$ we have

$$\frac{\lambda^3}{(\mathscr{D}^2+\lambda)^3}(x, x) \leqslant c\lambda^{1/2}\max\{B_L^*(x), \lambda\}. \tag{7.1}$$

For the proof of (4.7) we can use $n(XY^2X; c) = n(YX^2Y; c)$ for positive operators X, Y, to bring $|V-E|_-$ in the middle in (4.5), and estimate it by V, then undo it

$$(II) \leqslant \int_0^\infty n\left(V^{1/2}\frac{P^3}{(\mathscr{D}^2+P)^3}V^{1/2}, \frac{1}{16}E\right)\mathrm{d}E = 16P^3\,\mathrm{Tr}\left(V^{1/2}\frac{1}{(\mathscr{D}^2+P)^3}V^{1/2}\right). \tag{7.2}$$

Using (7.1) for $\lambda := P$, we immediately obtain (4.7). Using that $\Pi_{\tilde{\mathscr{D}}^2 \leqslant \lambda} \leqslant 8\lambda^3(\tilde{\mathscr{D}}^2+\lambda)^{-3}$ from the spectral theorem, we also obtain Corollary 2.2. Note that (2.4) holds for any $\lambda \geqslant 0$ since the case $\lambda < \lambda_*$ is trivial.

Proof of Lemma 7.1. We insert the partition of unity, recall the bounds (5.5) and use (5.9) to obtain

$$(\mathscr{D}^2+\lambda)^3 = \sum_{k\in\Lambda} [\mathscr{D}^3\theta_k^2\mathscr{D}^3 + 3\lambda\mathscr{D}^2\theta_k^2\mathscr{D}^2 + 3\lambda^2\mathscr{D}\theta_k^2\mathscr{D} + \lambda^3\theta_k^2]$$

$$\geqslant c \sum_{k\in\Lambda} [\mathscr{D}_k^3\theta_k^2\mathscr{D}_k^3 + \lambda\mathscr{D}_k^2\eta_k^2\mathscr{D}_k^2 + \lambda^2\mathscr{D}_k\omega_k^2\mathscr{D}_k + \lambda^2\mathscr{D}_k\theta_k^2\mathscr{D}_k + \lambda^3\chi_k^2]. \tag{7.3}$$

We use the $\lambda^2\mathscr{D}_k\theta_k^2\mathscr{D}_k$ term to save a derivative in the third direction. Let $\tilde{\mathscr{D}}_k$ be the Dirac operator constructed in Section 6 for the vector potential $\mathbf{A}_k$. Then there exists a spin rotation $U_k: \mathbf{R}^3 \mapsto SU(2)$ (see (6.12)) such that

$$\mathscr{D}_k = U_k\tilde{\mathscr{D}}_kU_k^*. \tag{7.4}$$

By Schwarz' inequality, $|\nabla_k\theta| \leqslant c(\varepsilon L)^{-1}$ and (6.18) from Lemma 6.2 applied to the operator $\tilde{\mathscr{D}}_k^2$ we have

$$\mathscr{D}_k\theta_k^2\mathscr{D}_k \geqslant c\theta_k\mathscr{D}_k^2\theta_k - c(\varepsilon L)^{-2}\,\eta_k \geqslant c\theta_kU_kD_{\mathbf{n}_k}^2U_k^*\theta_k - c\varepsilon^2P\eta_k.$$

We can sum up these inequalities, use (5.5) and $\lambda \geqslant P$, to obtain from (7.3) that for sufficiently small ε

$$(\mathscr{D}^2+\lambda)^3 \geqslant c\sum_k [\mathscr{D}_k^3\theta_k^2\mathscr{D}_k^3 + \lambda\mathscr{D}_k^2\eta_k^2\mathscr{D}_k^2 + \lambda^2\mathscr{D}_k\omega_k^2\mathscr{D}_k + \lambda^3\chi_k^2]$$

$$+ c\lambda^2\sum_k \theta_kU_k(D_{\mathbf{n}_k}^2+\lambda)\,U_k^*\theta_k.$$

Applying the estimate (6.19) from Proposition 6.2 to the operators $\tilde{\mathscr{D}}_k = U_k^*\mathscr{D}_kU_k$ we obtain that

$$(\mathscr{D}^2+\lambda)^3 \geqslant \sum_k \theta_kU_k((\delta_kT_k)^3 + \lambda^2D_{\mathbf{n}_k}^2 + \lambda^3)\,U_k^*\theta_k,$$

where $\delta_k := \min\{\lambda b_k^{-1}, 1\}$, the operators $\mathbf{D}_k$ and T_k are obtained by applying the construction of Section 6 for the vector potential $\mathbf{A}_k$.

Hence by the "Pull-up" formula (5.10) we get

$$\frac{1}{(\mathscr{D}^2+\lambda)^3} \leqslant c\sum_{k\in\Lambda} \theta_kU_k\,\frac{1}{(\delta_kT_k)^3 + \lambda^2D_{\mathbf{n}_k}^2 + \lambda^3}\,U_k^*\theta_k. \tag{7.5}$$

We apply (6.20) for the magnetic field $\mathbf{B}_k = \nabla \times \mathbf{A}_k$ satisfying the bound (6.2) with $b = cb_k$ from (5.7) to get

$$\frac{1}{(\mathscr{D}^2+\lambda)^3}(x,x) \leqslant c\,\lambda^{1/2} \sum_{k:\,\theta_k(x)\neq 0} \max\{b_k, \lambda\},$$

and use that only finitely many θ_k's are nonzero at any $x \in \mathbf{R}^3$, and for all such k's we have $b_k \leqslant B_L^*(x)$. This completes the proof of (7.1). ▮

8. PROOF OF THE POSITIVE ENERGY REGIME

In this section we prove (4.6). We start by applying the localization (5.11) in (4.4)

$$(I) \leqslant \int_0^\infty n\left(\sum_{k\in\Lambda} |V-E|_-^{1/2}\,\theta_k\,\frac{1}{\mathscr{D}_k^2+P+E}\,\theta_k\,|V-E|_-^{1/2}, \frac{1}{4}\right)\mathrm{d}E.$$

Since the supports of at most C of the functions θ_k can overlap (see (5.5)), we can "pull out" the summation at the expense of descreasing the energy threshold $\frac{1}{4}$ (see Section 6 of ref. 8 for more details). Therefore

$$(I) \leqslant \sum_{k\in\Lambda} \int_0^\infty n\left(|V-E|_-^{1/2}\,\theta_k\,\frac{1}{\mathscr{D}_k^2+P+E}\,\theta_k\,|V-E|_-^{1/2}, \frac{1}{4(C+1)}\right)\mathrm{d}E. \quad (8.1)$$

For any $\lambda \in \mathbf{R}$ let

$$\Pi_k(\lambda) := \chi(\mathscr{D}_k^2 \leqslant \lambda)$$

denote the spectral projection below the level λ in the spectrum of $\mathscr{D}_k^2$. We have

$$(I) \leqslant (I/1) + (I/2)$$

with

$$(I/1) := \sum_{k\in\Lambda} \int_0^\infty n\left(|V-E|_-^{1/2}\,\theta_k\,\frac{I-\Pi_k(b_k-P-E)}{\mathscr{D}_k^2+P+E}\,\theta_k\,|V-E|_-^{1/2}, \frac{1}{8(C+1)}\right)\mathrm{d}E, \quad (8.2)$$

$$(I/2) := \sum_{k\in\Lambda} \int_0^\infty n\left(|V-E|_-^{1/2}\,\theta_k\,\frac{\Pi_k(b_k-P-E)}{\mathscr{D}_k^2+P+E}\,\theta_k\,|V-E|_-^{1/2}, \frac{1}{8(C+1)}\right)\mathrm{d}E. \quad (8.3)$$

The estimate of the first term is easy:

$$\frac{I-\Pi_k(b_k-P-E)}{\mathscr{D}_k^2+P+E} \leqslant \frac{c}{\mathscr{D}_k^2+cb_k+E} \leqslant \frac{c}{(-i\nabla+\mathbf{A}_k)^2+E}$$

using the spectral theorem first, then the Lichnerowicz formula in the Euclidean coordinates:

$$\mathscr{D}_k^2=(-i\nabla+\mathbf{A}_k)^2+\boldsymbol{\sigma}\cdot\mathbf{B}_k$$

and $\boldsymbol{\sigma}\cdot\mathbf{B}_k \geqslant -cb_k$, where the constant is from (5.7). Then we can proceed as in the proof of the original Lieb–Thirring inequality[14] for each summand in (8.2):

$$\begin{aligned}
&\int_0^\infty n\left(|V-E|_-^{1/2}\,\theta_k\,\frac{c}{(-i\nabla+\mathbf{A}_k)^2+E}\,\theta_k\,|V-E|_-^{1/2},\frac{1}{8(C+1)}\right)\mathrm{d}E\\
&\quad\leqslant 64(C+1)^2\int_0^\infty \mathrm{Tr}\left(|V-E|_-^{1/2}\,\theta_k\,\frac{c}{(-i\nabla+\mathbf{A}_k)^2+E}\,\theta_k\,|V-E|_-^{1/2}\right)^2\mathrm{d}E\\
&\quad\leqslant c\int_0^\infty \mathrm{Tr}\left(|V-E|_-^2\,\theta_k^4\left[\frac{1}{(-i\nabla+\mathbf{A}_k)^2+E}\right]^2\right)\mathrm{d}E\\
&\quad\leqslant c\int_0^\infty\int |V(x)-E|_-^2\,\theta_k^4(x)\left[\frac{1}{-\Delta+E}\right]^2(x,x)\,\mathrm{d}x\,\mathrm{d}E\\
&\quad\leqslant c\int V^{5/2}\chi_k,
\end{aligned}$$

where we also used the diamagnetic inequality for the square of the resolvent. After the summation over k and using (5.5), we obtain the second term in the estimate (4.6).

For the term (I/2), we first notice that the dE integration can be restricted to $E \leqslant b_k$, and we can also assume that that $P \leqslant b_k$. After this restriction we estimate $\Pi_k(b_k-P-E)$ by $\Pi_k(b_k)$ and we obtain

$$\begin{aligned}
(I/2)&\leqslant \sum_{k\in\Lambda}\int_0^{b_k} n\left(|V-E|_-^{1/2}\,\theta_k\,\frac{\Pi_k(b_k)}{\mathscr{D}_k^2+P+E}\,\theta_k\,|V-E|_-^{1/2},\frac{1}{8(C+1)}\right)\mathrm{d}E\\
&\leqslant 8(C+1)\sum_{k\in\Lambda}\int_0^{b_k}\mathrm{Tr}\left(|V-E|_-^{1/2}\,\theta_k\,\frac{\Pi_k(b_k)}{\mathscr{D}_k^2+P+E}\,\theta_k\,|V-E|_-^{1/2}\right)\mathrm{d}E\\
&\leqslant c\sum_{k\in\Lambda}\int_0^{b_k}\int_{\mathbf{R}^3}|V(x)-E|_-\,\theta_k^2(x)\left[\int_0^\infty\frac{\chi(\lambda\leqslant b_k)}{\lambda+P+E}\,\mathrm{d}\Pi_k(\lambda)\right](x,x)\,\mathrm{d}x\,\mathrm{d}E,
\end{aligned} \tag{8.4}$$

using the spectral resolution of $\mathscr{D}_k^2$. Clearly

$$\int_0^\infty \frac{\chi(\lambda \leqslant b_k)}{\lambda + P + E}\, \mathrm{d}\Pi_k(\lambda)$$
$$= \frac{1}{b_k+P+E}\Pi_k(b_k) - \frac{1}{P+E}\Pi_k(0) + \int_0^{b_k} \frac{1}{(\lambda+P+E)^2}\Pi_k(\lambda)\,\mathrm{d}\lambda$$
$$\leqslant \frac{1}{b_k+E}\Pi_k(b_k) + \frac{P}{(P+E)^2}\Pi_k(P) + \int_P^{b_k} \frac{1}{(\lambda+E)^2}\Pi_k(\lambda)\,\mathrm{d}\lambda. \tag{8.5}$$

The conclusion of Corollary 2.2 also applies to $\mathscr{D}_k$, replacing the variation lengthscale $L_{\mathbf{n}}$ with $\varepsilon L_{\mathbf{n}}$, see (5.8), therefore

$$\Pi_k(\lambda)(x,x) \leqslant c\lambda^{1/2}\max\{B_L^*(x), \lambda\} \tag{8.6}$$

for $\lambda \geqslant \varepsilon^{-6}L^{-2}$, but then the same inequality also holds for all $\lambda \geqslant P = \varepsilon^{-4}L^{-2}$ if c is changed to $c\varepsilon^{-3}$ using the monotonicity of $\lambda \mapsto \Pi_k(\lambda)$.

From (8.5), (8.6) and the fact that $P \leqslant b_k \leqslant B_L^*(x)$ for $x \in \mathrm{supp}(\theta_k)$, we obtain

$$\left[\int_0^\infty \frac{\chi(\lambda \leqslant b_k)}{\lambda+P+E}\,\mathrm{d}\Pi_k(\lambda)\right](x,x) \leqslant c\varepsilon^{-3}B_L^*(x)\left(\frac{b_k^{1/2}}{b_k+E} + \frac{P^{3/2}}{(P+E)^2} + \int_P^{b_k}\frac{\lambda^{1/2}\mathrm{d}\lambda}{(\lambda+E)^2}\right)$$
$$\leqslant c\varepsilon^{-3}E^{-1/2}B_L^*(x).$$

Therefore we can complete the estimate of $(I/2)$ from (8.4)

$$(I/2) \leqslant c\varepsilon^{-3}\sum_{k\in\Lambda}\int_0^\infty\int_{\mathbf{R}^3} B_L^*(x)\,\theta_k^2(x)\,|V(x)-E|_-\,E^{-1/2}\,\mathrm{d}x\,\mathrm{d}E = c\varepsilon^{-3}\int B_L^* V^{3/2}.$$

This completes the proof of (4.6) after choosing ε to be a sufficiently small universal constant. ∎

1. APPENDIX: PROOF OF THE TECHNICAL LEMMAS

A.1. Proof of Lemma 5.1

Let $\mathscr{P}_k$ be the plane through $k \in \Lambda$ and orthogonal to $\mathbf{n}(k)$. We define $\tilde{\mathbf{n}}_k := \chi_k\mathbf{n} + (1-\chi_k)\,\mathbf{n}(k)$. Since $\tilde{\mathbf{n}}_k - \mathbf{n}(k) = (\mathbf{n}-\mathbf{n}(k))\,\chi_k = O(\varepsilon)$ from (2.2), we have $\tilde{\mathbf{n}}_k \cdot \mathbf{n}(k) \geqslant 1 - O(\varepsilon)$ and $|\tilde{\mathbf{n}}_k| = 1 + O(\varepsilon)$, in particular $\tilde{\mathbf{n}}_k \neq 0$ for sufficiently small ε. Moreover $\nabla\tilde{\mathbf{n}}_k = \chi_k\nabla\mathbf{n} + (\mathbf{n}-\mathbf{n}(k))\,\nabla\chi_k$, therefore $\nabla\tilde{\mathbf{n}}_k = O(L^{-1})$ since $|\mathbf{n}-\mathbf{n}(k)| = O(\varepsilon)$ on the support of χ_k from (2.2) and $L \leqslant L_{\mathbf{n}}$. Similar bounds hold for the higher derivatives. We define $\mathbf{n}_k := \tilde{\mathbf{n}}_k/|\tilde{\mathbf{n}}_k|$ and (5.8) can be easily checked.

Now $\mathbf{B}_k$ is defined as $\mathbf{B}_k := f_k \mathbf{n}_k$ where the function f_k is constructed as follows. We define

$$f_k := \chi_k \, |\mathbf{B}| + (1 - \chi_k) \, |\mathbf{B}(k)| \qquad \text{on } \mathscr{P}_k,$$

then clearly $|f_k| \leqslant cb_k$ on $\mathscr{P}_k$.

Since the vectorfield $\mathbf{n}_k$ is nearly parallel (clearly $\mathbf{n}_k \cdot \mathbf{n}(k) \geqslant 1 - O(\varepsilon)$), its integral curves define a transversal foliation to the plane $\mathscr{P}_k$. Hence we can extend f_k from $\mathscr{P}_k$ onto the whole $\mathbf{R}^3$ by ensuring that $\mathbf{B}_k$ is divergence free, i.e., $\operatorname{div} \mathbf{B}_k = f_k \operatorname{div} \mathbf{n}_k + \nabla_{\mathbf{n}_k} f_k = 0$. This requires integrating the equation

$$\nabla_{\mathbf{n}_k} (\log f_k) = -\operatorname{div} \mathbf{n}_k$$

along the integral curves of $\mathbf{n}_k$. Since $|\nabla \mathbf{n}_k| \leqslant cL^{-1}$, and it vanishes outside of $\Omega_k^{\#}$, we obtain $|f_k| + L \, |\nabla (f_k \mathbf{n}_k)| \leqslant cb_k$ everywhere as required in (5.7).

The original field strength $|\mathbf{B}|$ also satisfies the equation

$$\nabla_{\mathbf{n}} (\log |\mathbf{B}|) = -\operatorname{div} \mathbf{n},$$

and since $\mathbf{n}_k = \mathbf{n}$ on Ω_k by definition and $|\mathbf{B}| = f_k$ on $\mathscr{P}_k \cap \Omega_k^*$, we obtain that $f_k = |\mathbf{B}|$ on Ω_k. Therefore $\mathbf{B}_k \equiv \mathbf{B}$ on Ω_k^*.

This completes the definition of the extension $\mathbf{B}_k := f_k \mathbf{n}_k$ of the field $\mathbf{B}$ from Ω_k^* to the whole space together with the estimates on the field strength (5.7) and on the variation of the field direction (5.8).

Finally, the appropriate vector potential $\mathbf{A}_k$ is defined as $\mathbf{A}_k := \mathbf{A} + \mathbf{A}_k^{\#}$, where $\mathbf{A}_k^{\#}$ is the Poincaré gauge of the magnetic field $\mathbf{B}_k - \mathbf{B}$ centered at k. Since $\mathbf{B}_k \equiv \mathbf{B}$ on Ω_k^* and Ω_k^* is convex, we have $\mathbf{A}_k^{\#} \equiv 0$ on Ω_k^* as well. This completes the proof of Lemma 5.1. ∎

A.2. Proof of Lemma 6.1

Using (6.10) and $D_{\mathbf{n}} = D_3$, we compute the commutator:

$$\begin{aligned}
&[\mathbf{D}^t Z \mathbf{D}, D_3] \\
&\quad = [D_j Z_{jk} D_k, D_3] = D_j Z_{jk} [D_k, D_3] + i D_j (\partial_3 Z_{jk}) \, D_k + [D_j, D_3] \, Z_{jk} D_k \\
&\quad = (-i) \Bigg(D_j Z_{jk} \left(Q_{k3}^m D_m - \frac{i}{2} (\partial_m Q_{k3}^m) + \beta_{k3} \right) \\
&\qquad + \left(Q_{j3}^m D_m - \frac{i}{2} (\partial_m Q_{j3}^m) + \beta_{j3} \right) Z_{jk} D_k - D_j (\partial_3 Z_{jk}) \, D_k \Bigg) \\
&\quad = (-i) \Bigg(D_j Z_{jk} Q_{k3}^m D_m - \frac{i}{2} (\partial_m Q_{j3}^m)(Z_{kj} - Z_{jk}) \, D_k - \frac{1}{2} \partial_j (Z_{jk} (\partial_m Q_{k3}^m)) \\
&\qquad + Z_{jk} \beta_{k3} D_j - i \partial_j (Z_{jk} \beta_{k3}) + (D_m Q_{j3}^m + \beta_{j3}) \, Z_{jk} D_k - D_j (\partial_3 Z_{jk}) \, D_k \Bigg),
\end{aligned}$$

and

$$[h, D_3] = i\partial_3 h.$$

Now we match the coefficients in

$$[\mathbf{D}^t Z \mathbf{D} + h, D_3] = 0.$$

Matching the highest (second) order terms gives

$$\partial_3 Z = ZQ + Q^t Z$$

with the matrix $Q_{ab} := Q^b_{a3}$.

On the supporting plane we choose $Z \equiv I$. Using (6.4) we have $\nabla^\gamma Q = O(\varepsilon^{-\gamma} L^{-1-\gamma})$, $0 \leqslant \gamma \leqslant 4$, in a neighborhood of size $O(\varepsilon L)$, otherwise $Q \equiv 0$, hence we obtain the solution Z satisfying (6.14), and Z is clearly a symmetric, real matrix.

Using the established symmetry of Z, to match the first order terms, we need that

$$Z_{kj}\beta_{j3} = 0 \qquad k = 1, 2, 3$$

but clearly $\beta_{j3} = 0$ because $e_3 = \mathbf{n}$ was the direction of the magnetic field.

Finally the constant term can be matched by choosing h to be 0 on the supporting plane and such that h satisfy

$$\partial_3 h = -\tfrac{1}{2}\,\partial_j (Z_{jk}(\partial_m Q^m_{k3})).$$

Using again that $\nabla^\gamma Q = O(\varepsilon^{-\gamma} L^{-1-\gamma})$, $0 \leqslant \gamma \leqslant 4$, in a neighborhood of size $O(\varepsilon L)$, otherwise $Q \equiv 0$, we obtain that the solution satisfies (6.15). ∎

A.3. Proof of Lemma 6.2

Proof of (6.18). We use $\tilde{\mathcal{D}} = \boldsymbol{\sigma} \cdot \boldsymbol{\Pi}$, (6.7), the self-adjointness of D_j, (6.4) and a Schwarz' inequality to obtain

$$\tilde{\mathcal{D}}^2 = \left(\sum_{j=1}^{3} \sigma^j \left[D_j + \frac{i}{2}\,\mathrm{div}\, e_j + \frac{1}{2}\,\sigma \cdot \boldsymbol{\omega}(e_j) \right] \right)^2 \geqslant \frac{1}{2} \left(\sum_{j=1}^{3} \sigma^j D_j \right)^2 - O(L^{-2}). \quad \text{(A.1)}$$

Furthermore

$$\left(\sum_{j=1}^{3} \sigma^j D_j \right)^2 = (\sigma_\perp \cdot D_\perp)^2 + D_3^2 + \sum_{j=1}^{2} \{\sigma^j D_j, \sigma^3 D_3\}, \quad \text{(A.2)}$$

where $\sigma_\perp \cdot D_\perp := \sigma^1 D_1 + \sigma^2 D_2$ and $\{X, Y\} = XY + YX$ denotes the anti-commutator. Using that $\sigma^1\sigma^3 = -i\sigma^2$ and $\sigma^2\sigma^3 = i\sigma^1$, we obtain from (6.10) and $\beta(e_3, e_j) = 0$, $j = 1, 2$, that

$$\begin{aligned}\{\sigma_\perp \cdot D_\perp, \sigma^3 D_3\} &= i\sigma^2[D_3, D_1] - i\sigma^1[D_3, D_2] \\ &= \sum_{a,b=1}^{2} \sigma^a M_{ab} D_b + D_b M_{ab} \sigma^a + \frac{1}{2}\sum_{a=1}^{2} (-1)^a \sigma^a (Q^3_{3\bar{a}} D_3 + D_3 Q^3_{3\bar{a}}) \qquad \text{(A.3)}\end{aligned}$$

with $M_{ab} := \frac{1}{2}(-1)^a Q^b_{3,\bar{a}}$, $a, b = 1, 2$, where $\bar{1} := 2$ and $\bar{2} := 1$. We use Schwarz' inequality and that $Q^\ell_{jk} = O(L^{-1})$ to estimate the last terms containing D_3. In summary, we obtain from (A.1)–(A.3) that

$$\tilde{\mathscr{D}}^2 \geqslant \tfrac{1}{2}[(\sigma_\perp \cdot D_\perp)^2 + \tfrac{1}{2} D_3^2 + \sigma^a M_{ab} D_b + D_b M_{ab} \sigma^a] - O(L^{-2}). \qquad \text{(A.4)}$$

We compute a two dimensional Lichnerowicz formula

$$\begin{aligned}(\sigma_\perp \cdot D_\perp)^2 &= D_1^2 + D_2^2 + i\sigma^3[D_1, D_2] \\ &= D_1^2 + D_2^2 + B\sigma^3 + \tfrac{1}{2}\sigma^3(Q^m_{12} D_m + D_m Q^m_{12})\end{aligned}$$

using that $\beta_{12} = B$. Let $P_\pm := \frac{1}{2}(1 \pm \sigma^3)$ and note that P_-, P_+ are orthogonal projections commuting with σ^3, and $P_\pm \sigma^a = \sigma^a P_\mp$ for $a = 1, 2$. In particular $(\sigma_\perp \cdot D_\perp)^2$ commutes with $P_\pm$. Therefore

$$\begin{aligned}(\sigma_\perp \cdot D_\perp)^2 &= P_+(\sigma_\perp \cdot D_\perp)^2 P_+ + P_-(\sigma_\perp \cdot D_\perp)^2 P_- \\ &\geqslant P_+(\sigma_\perp \cdot D_\perp)^2 P_+\end{aligned}$$

and

$$\begin{aligned}&P_+(\sigma_\perp \cdot D_\perp)^2 P_+ \\ &\quad = P_+(D_1^2 + D_2^2 + B) P_+ + \tfrac{1}{2} P_+ \sigma^3 (Q^m_{12} D_m + D_m Q^m_{12}) P_+ \\ &\quad \geqslant P_+(D_1^2 + D_2^2 + B) P_+ - \tfrac{1}{4} P_+(D_1^2 + D_2^2 + D_3^2) P_+ - O(L^{-2})\end{aligned}$$

by a Schwarz' inequality and (6.4). We obtain from (A.4) that

$$\mathscr{D}^2 \geqslant \tfrac{1}{8}[P_+(D_1^2 + D_2^2 + B) P_+ + D_3^2] + \tfrac{1}{2}[\sigma^a M_{ab} D_b + D_b M_{ab} \sigma^a] - O(L^{-2}). \qquad \text{(A.5)}$$

Finally, we compute

$$\begin{aligned}&\sigma^a M_{ab} D_b + D_b M_{ab} \sigma^a \\ &\quad = P_+[\sigma^a M_{ab} D_b + D_b M_{ab} \sigma^a] P_- + P_-[\sigma^a M_{ab} D_b + D_b M_{ab} \sigma^a] P_+ \\ &\quad = 2P_- \sigma^a M_{ab} D_b P_+ + 2P_+ D_b M_{ab} \sigma^a P_- + P_+ \sigma^a [M_{ab}, D_b] P_- + P_-[D_b, M_{ab}] \sigma^a P_+.\end{aligned}$$

By a Schwarz' inequality and (6.4) we obtain

$$\sigma^a M_{ab} D_b + D_b M_{ab} \sigma^a \geqslant -\tfrac{1}{4} P_+ (D_1^2 + D_2^2) P_+ - O(\varepsilon^{-1} L^{-2}),$$

which, combined with (A.5) gives (6.18) since $D_3 = D_{\mathbf{n}}$.

Proof of (6.19). From the Lichnerowicz formula (6.13) and (6.7) we have

$$\tilde{\mathscr{D}}^2 = \left(D_j - \frac{i}{2} \operatorname{div} e_j + \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j) \right)\left(D_j + \frac{i}{2} \operatorname{div} e_j + \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j) \right) + \sigma(\bigstar \beta)$$

$$= \mathbf{D}^2 + \frac{1}{2} [\boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j) D_j + D_j \boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j)] + \mathscr{E}, \tag{A.6}$$

where $\mathscr{E}$ is a zeroth order error that satisfies

$$|\mathscr{E}| + L |\nabla \mathscr{E}| \leqslant (P + b) \tag{A.7}$$

from (6.4) and (6.2).

Lemma 1.1. The following comparison estimates hold

$$\tfrac{1}{2} \mathbf{D}^2 - c(P+b) \leqslant \tilde{\mathscr{D}}^2 \leqslant 2\mathbf{D}^2 + c(P+b), \tag{A.8}$$

$$\tfrac{1}{4} (\mathbf{D}^2)^2 - c(P+b)^2 \leqslant \tilde{\mathscr{D}}^4 \leqslant 4(\mathbf{D}^2)^2 + c(P+b)^2, \tag{A.9}$$

$$\tfrac{1}{4} (\mathbf{D}^2)^3 - c(P+b)^3 \leqslant \tilde{\mathscr{D}}^6 \leqslant 4(\mathbf{D}^2)^3 + c(P+b)^3. \tag{A.10}$$

Proof. The proof of (A.8) is straightforward by a Schwarz' inequality and estimating the geometric terms as

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j))^2 \leqslant |\boldsymbol{\omega}(e_j)|^2 \leqslant \varepsilon^2 P \tag{A.11}$$

from (6.4). For (A.9) we again use a Schwarz' inequality

$$\tilde{\mathscr{D}}^4 \leqslant 3(\mathbf{D}^2)^2 + 3[D_j (\boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j))^2 D_j + \boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j) D_j^2 \boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j)] + 3(P+b)^2.$$

The first term in the square bracket is bounded by $\varepsilon^2 P \mathbf{D}^2$ using (A.11). In the second term we commute through and obtain

$$\boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j) D_j^2 \boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j) \leqslant 2 D_j (\boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j))^2 D_j + 2 |\partial_j \boldsymbol{\sigma} \cdot \boldsymbol{\omega}(e_j)|^2 \leqslant 2\varepsilon^2 P \mathbf{D}^2 + \varepsilon^4 P^2$$

from (6.4). Altogether, with a further Schwarz' inequality we obtain the upper bound in (A.9). A similar argument gives a lower bound as well.

Finally, to bound the sixth power from below we use (A.8)

$$\tilde{\mathscr{D}}^6 \geqslant \tilde{\mathscr{D}}^2 (\tfrac{1}{2} \mathbf{D}^2 - c(P+b)) \tilde{\mathscr{D}}^2. \tag{A.12}$$

For the first term we use (A.6) and Schwarz

$$\tilde{\mathscr{D}}^2\mathbf{D}^2\tilde{\mathscr{D}}^2 \geqslant \tfrac{1}{2}(\mathbf{D}^2)^3 - c\boldsymbol{\sigma}\cdot\boldsymbol{\omega}(e_j)\, D_j\mathbf{D}^2D_j\boldsymbol{\sigma}\cdot\boldsymbol{\omega}(e_j)$$
$$- cD_j\boldsymbol{\sigma}\cdot\boldsymbol{\omega}(e_j)\,\mathbf{D}^2\boldsymbol{\sigma}\cdot\boldsymbol{\omega}(e_j)\,D_j - c\mathscr{E}\mathbf{D}^2\mathscr{E}.$$

All the terms $\boldsymbol{\sigma}\cdot\boldsymbol{\omega}(e_j)$ can be commuted in the middle and estimated by εP from (6.4). Similarly, D_j's can be commuted in the middle to estimate

$$\varepsilon PD_j\mathbf{D}^2D_j \leqslant 2\varepsilon P(\mathbf{D}^2)^2 + \varepsilon P^2\mathbf{D}^2 + \varepsilon P(P+b)^2$$

using (6.10). The resulting terms of the form $\varepsilon P(\mathbf{D}^2)^2$ and $\varepsilon P^2\mathbf{D}^2$ can be controlled by $2\varepsilon(\mathbf{D}^2)^3 + c\varepsilon P^3$. The last term in (A.12) can be dealt with in the same way after estimating $\tilde{\mathscr{D}}^4$ by $(\mathbf{D}^2)^2$ from (A.9). For the term $\mathscr{E}\mathbf{D}^2\mathscr{E}$ we use

$$\mathscr{E}\mathbf{D}^2\mathscr{E} \leqslant 2D_j\mathscr{E}^2D_j + 2\,|\nabla\mathscr{E}|^2$$

and the bound (A.7).

Putting all these estimates together and applying several Schwarz' inequality, we easily arrive at the lower bound in (A.10). The upper bound can be proven similarly, although we will not need it. ∎

After these preparations, now turn to the actual proof of (6.19).

$$\tilde{\mathscr{D}}^3\theta^2\tilde{\mathscr{D}}^3 \geqslant \tfrac{1}{2}\,\tilde{\mathscr{D}}^2\theta\tilde{\mathscr{D}}^2\theta\tilde{\mathscr{D}}^2 - 2\tilde{\mathscr{D}}^2\,|\nabla\theta|^2\,\tilde{\mathscr{D}}^2 \geqslant \tfrac{1}{2}\,\tilde{\mathscr{D}}^2\theta\tilde{\mathscr{D}}^2\theta\tilde{\mathscr{D}}^2 - \varepsilon\lambda\tilde{\mathscr{D}}^2\eta^2\tilde{\mathscr{D}}^2. \qquad \text{(A.13)}$$

The error can be absorbed into the second term on the left of (6.19). We continue with the main term

$$\tilde{\mathscr{D}}^2\theta\tilde{\mathscr{D}}^2\theta\tilde{\mathscr{D}}^2 \geqslant \frac{\delta}{4}\,\tilde{\mathscr{D}}\theta\tilde{\mathscr{D}}^4\theta\tilde{\mathscr{D}} - \delta\tilde{\mathscr{D}}\sigma(\mathrm{d}\theta)\,\tilde{\mathscr{D}}^2\sigma(\mathrm{d}\theta)\,\tilde{\mathscr{D}}. \qquad \text{(A.14)}$$

In the error term we use (A.8) and $|\nabla\theta|^2 \leqslant \varepsilon P\omega^2$

$$\delta\tilde{\mathscr{D}}\sigma(\mathrm{d}\theta)\,\tilde{\mathscr{D}}^2\sigma(\mathrm{d}\theta)\,\tilde{\mathscr{D}} \leqslant c\,\delta\tilde{\mathscr{D}}\sigma(\mathrm{d}\theta)\,\mathbf{D}^2\sigma(\mathrm{d}\theta)\,\tilde{\mathscr{D}} + \varepsilon\,\delta P(P+b)\,\tilde{\mathscr{D}}\omega^2\tilde{\mathscr{D}}. \qquad \text{(A.15)}$$

The second term can be absorbed into the third term on the left side of (6.19) since $\delta(P+b) \leqslant 2\lambda$ and $P \leqslant \lambda$. In the first term in (A.15) we can eliminate the σ's using a Schwarz' inequality, $(\partial_j\theta_j)\,\sigma^j\,\mathbf{D}^2\sigma^k(\partial_k\theta) \leqslant 3(\partial_j\theta_j)\,\mathbf{D}^2(\partial_j\theta_j)$, and the fact that $\mathbf{D}^2$ is identity in the spin space, hence it commutes with the Pauli matrices. After that we can again use (A.8) and commute $\nabla\theta$ inside

$$\begin{aligned}\delta\tilde{\mathscr{D}}\sigma(\mathrm{d}\theta)\,\mathbf{D}^2\sigma(\mathrm{d}\theta)\,\tilde{\mathscr{D}} &\leqslant c\,\delta\tilde{\mathscr{D}}(\partial_j\theta)\,\mathbf{D}^2(\partial_j\theta)\,\tilde{\mathscr{D}}\\ &\leqslant c\,\delta\tilde{\mathscr{D}}(\partial_j\theta)(\tilde{\mathscr{D}}^2+P+b)(\partial_j\theta)\,\tilde{\mathscr{D}}\\ &\leqslant c\,\delta\tilde{\mathscr{D}}^2\,|\nabla\theta|^2\,\tilde{\mathscr{D}}^2+c\,\delta\tilde{\mathscr{D}}\,|\nabla^2\theta|^2\,\tilde{\mathscr{D}}+c\,\delta\varepsilon P(P+b)\,\tilde{\mathscr{D}}\omega^2\tilde{\mathscr{D}}\\ &\leqslant c\varepsilon\lambda\tilde{\mathscr{D}}^2\eta^2\tilde{\mathscr{D}}^2+c\varepsilon\lambda^2\tilde{\mathscr{D}}\omega^2\tilde{\mathscr{D}}, \qquad \text{(A.16)}\end{aligned}$$

and these terms can be absorbed into the lower order terms on the left side of (6.19).

Finally, we continue the estimate of the main term in (A.14):

$$\frac{\delta}{4}\,\tilde{\mathscr{D}}\theta\tilde{\mathscr{D}}^4\theta\tilde{\mathscr{D}} \geqslant \frac{\delta^2}{4}\,\tilde{\mathscr{D}}\theta\tilde{\mathscr{D}}^4\theta\tilde{\mathscr{D}} \geqslant \frac{\delta^2}{8}\,\theta\tilde{\mathscr{D}}^6\theta-\delta^2\sigma(\mathrm{d}\theta)\,\tilde{\mathscr{D}}^4\sigma(\mathrm{d}\theta). \qquad \text{(A.17)}$$

In the first term we use (A.10) after the estimate $\delta^2\geqslant\delta^3$. In the error term we change $\tilde{\mathscr{D}}^4$ to $(\mathbf{D}^2)^2$ using the upper bound in (A.9), then we can eliminate the σ's, use the lower bound in (A.9) to change $(\mathbf{D}^2)^2$ back to $\tilde{\mathscr{D}}^4$ and finally commute $\partial\theta$'s in the middle similarly to (A.16). All the error terms can be absorbed into the lower order terms on the left side of (6.19). We omit the details.

Putting all these estimates together, we obtain for sufficiently small ε

$$\tilde{\mathscr{D}}^3\theta^2\tilde{\mathscr{D}}^3+\lambda\tilde{\mathscr{D}}^2\eta^2\tilde{\mathscr{D}}^2+\lambda^2\tilde{\mathscr{D}}\omega^2\tilde{\mathscr{D}}+\lambda^3\chi^2\geqslant c\,\delta^3\theta(\mathbf{D}^2)^3\,\theta,$$

Since $T=\mathbf{D}\cdot Z\mathbf{D}+h$ and Z, h satisfy (6.14), (6.15), in particular Z is close to the identity, it is a trivial exercise with commutators to verify that

$$(\mathbf{D}^2)^3\geqslant cT^3-\varepsilon P^3$$

and the error can be absorbed into $\lambda^3\chi^2$. This completes the proof of (6.19). ∎

A.4. Proof of Lemma 6.3

Since $X:=\delta\lambda^{-1}T$ and $Y:=\lambda^{-1}D_{\mathbf{n}}^2$ commute, we can estimate

$$\frac{\lambda^3}{(\delta T)^3+\lambda^2D_{\mathbf{n}}^2+\lambda^3}\leqslant\frac{c}{(X+Y^{1/3}+I)^3}\leqslant c\int_0^\infty t^2e^{-t}e^{-tX}e^{-tY^{1/3}}\,\mathrm{d}t,$$

therefore

$$\frac{\lambda^3}{(\delta T)^3+\lambda^2 D_{\mathbf{n}}^2+\lambda^3}(x,x)\leqslant c\int_0^\infty t^2 e^{-t}\,|e^{-tX}(x,y)|\,|e^{-tY^{1/3}}(y,x)|\,\mathrm{d}y\,\mathrm{d}t. \quad \text{(A.18)}$$

We recall that $\hat{T}$ and $\hat{D}_j$ refer to the nonmagnetic counterparts of T and D_j, we also denote $\hat{X}:=\delta\lambda^{-1}\hat{T}$, $\hat{Y}:=\lambda^{-1}\hat{D}_{\mathbf{n}}^2$. The operator T is a second order uniformly elliptic operator, its heat kernel has a Feynman–Kac representation with an imaginary term in the exponent due to α. Following the standard proof of the diamagnetic inequality using Feynman–Kac formula (see, e.g., 18), we can estimate the oscillatory term trivially by one and we obtain

$$|e^{-tX}(x,y)|\leqslant e^{-t\hat{X}}(x,y).$$

From standard off-diagonal elliptic heat kernel estimates (see, e.g., 15) we obtain

$$e^{-t\hat{X}}(x,y)\leqslant c(t\,\delta\lambda^{-1})^{-3/2}\exp\left(-\frac{c(x-y)^2}{t\,\delta\lambda^{-1}}\right). \quad \text{(A.19)}$$

The operator $D_{\mathbf{n}}$ and $\hat{D}_{\mathbf{n}}$ are conjugated by a phase factor

$$D_{\mathbf{n}}=e^{-i\phi}\,\hat{D}_{\mathbf{n}}e^{i\phi},$$

where ϕ is a solution of $\partial_{\mathbf{n}}\phi=\alpha(\mathbf{n})$. Since the integral curves of $\mathbf{n}$ are disjoint, a solution ϕ exists globally. Therefore

$$Y=e^{-i\phi}\hat{Y}e^{i\phi}$$

hence, by spectral theorem,

$$|e^{-tY^{1/3}}(y,x)|=|e^{-t\hat{Y}^{1/3}}(y,x)|.$$

Since $\hat{Y}$ is a one-dimensional Laplacian, its spectral decomposition is exactly computable in appropriate coordinates (ξ_1,ξ_2,ξ_3) chosen as follows. Let ξ_3 be the arclength parameter along the integral curves of $\mathbf{n}$ with $\xi_3\equiv 0$ on the supporting plane $\mathscr{P}$. We choose an arbitrary Euclidean coordinatization $\xi_\perp:=(\xi_1,\xi_2)$ on $\mathscr{P}$. Since the integral curves give a regular foliation of $\mathbf{R}^3$, we can extend the coordinates ξ_1,ξ_2 onto $\mathbf{R}^3$ by setting it constant along each integral curve.

Since the integral curves are straight lines outside of $\Omega^{\#}$, and they are almost parallel inside, it is easy to see that

$$c_1\leqslant\frac{|\xi(x)-\xi(y)|}{|x-y|}\leqslant c_2 \quad \text{(A.20)}$$

for any x, y with universal constants c_1, c_2. A detailed proof of this obvious fact can be obtained along the lines of the proof of Lemma 8.4 of ref. 8, therefore we omit it here. Moreover, the Jacobian determinant $\det(D\xi(x))$ of the transformation $x \mapsto \xi(x)$ is universally bounded and likewise for the inverse map.

Define the unitary transform $U: L^2(\mathbf{R}^3, \mathrm{d}\xi) \to L^2(\mathbf{R}^3, \mathrm{d}x)$

$$[U\psi](x) = |\det(D\xi(x))|^{1/2}\, \psi(\xi(x)).$$

We then have that

$$U^* \hat{D}_{\mathbf{n}} U = -i\partial_{\xi_3}.$$

Hence

$$\begin{aligned} e^{-t\hat{Y}^{1/3}}(x, y) &= |\det(D\xi(x))|^{1/2}\, |\det(D\xi(y))|^{1/2}\, e^{-t[(-i\partial_{\xi_3})^2]^{1/3}}(\xi(x), \xi(y)) \\ &= |\det(D\xi(x))|^{1/2}\, |\det(D\xi(y))|^{1/2}\, \delta(\xi_\perp(x), \xi_\perp(y)) \\ &\quad \times \int_{\mathbf{R}} e^{-t\lambda^{-1/3}|p|^{2/3}} e^{ip(\xi_3(x)-\xi_3(y))}\, \mathrm{d}p. \end{aligned}$$

Hence

$$|e^{-t\hat{Y}^{1/3}}(x, y)| \leqslant c\lambda^{1/2} t^{-3/2}\, \delta(\xi_\perp(x), \xi_\perp(y)). \tag{A.21}$$

Now we can estimate the diagonal kernel (A.18), using (A.19), (A.21), (A.20), and the boundedness of the Jacobian determinants;

$$\begin{aligned} &\frac{\lambda^3}{(\delta T)^3 + \lambda^2 D_{\mathbf{n}}^2 + \lambda^3}(x, x) \\ &\leqslant c\lambda^2 \delta^{-3/2} \int_0^\infty \int_{\mathbf{R}^3} t^{-1} e^{-t} \exp\left(-\frac{c(\xi(x)-\eta)^2}{t\,\delta\lambda^{-1}}\right) \delta(\eta_\perp, \xi_\perp(x))\, \mathrm{d}\eta\, \mathrm{d}t \\ &= c\lambda^2 \delta^{-3/2} \int_0^\infty \int_{\mathbf{R}} t^{-1} e^{-t} \exp\left(-\frac{c(\xi_3(x)-\eta_3)^2}{t\delta\lambda^{-1}}\right) \mathrm{d}\eta_3\, \mathrm{d}t \\ &= c\lambda^{3/2}\delta^{-1}. \end{aligned} \tag{A.22}$$

This completes the proof of (6.20). ▮

ACKNOWLEDGMENTS

The work of L.E. was partially supported by NSF Grant DMS-0200235 and by MSRI. The work of J.P.S. was partially supported by the

Danish Natural Science Research Council, by the EU research network HPRN-CT-2002-00277, and by the Institute for advanced study through NSF grant DMS-0111298. Both L.E. and J.P.S. were partially supported by MSRI and by MaPhySto—A Network in Mathematical Physics and Stochastics, funded by The Danish National Research Foundation.

REFERENCES

1. L. Bugliaro, C. Fefferman, J. Fröhlich, G. M. Graf, and J. Stubbe, A Lieb–Thirring bound for a magnetic Pauli Hamiltonian, *Comm. Math. Phys.* **187**:567–582 (1997).
2. L. Bugliaro, J. Fröhlich, and G. M. Graf, Stability of quantum electrodynamics with nonrelativistic matter, *Phys. Rev. Lett.* **77**:3494–3497 (1996).
3. L. Bugliaro, C. Fefferman, and G. M. Graf, A Lieb–Thirring bound for a magnetic Pauli Hamiltonian, II, *Rev. Mat. Iberoamericana* **15**:593–619 (1999).
4. L. Erdős, Magnetic Lieb–Thirring inequalities, *Comm. Math. Phys.* **170**:629–668 (1995).
5. L. Erdős and J. P. Solovej, Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields. I. Non-asymptotic Lieb–Thirring type estimate, *Duke J. Math.* **96**:127–173 (1999).
6. L. Erdős and J. P. Solovej, Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields. II. Leading order asymptotic estimates, *Comm. Math. Phys.* **188**:599–656 (1997).
7. L. Erdős and J. P. Solovej, The kernel of Dirac operators on S^3 and $\mathbf{R}^3$, *Rev. Math. Phys.* **13**:1247–1280 (2001).
8. L. Erdős and J. P. Solovej, Uniform Lieb–Thirring inequality for the three dimensional Pauli operator with a strong non-homogeneous magnetic field. Available at http://xxx.lanl.gov/pdf/math-ph/0304017. to appear in *Ann. Henri Poincaré* (2003).
9. E. H. Lieb, *Lieb–Thirring Inequalities*, Kluwer Encyclopedia of Mathematics, Supplement, Vol. II (2000), pp. 311–313.
10. E. H. Lieb, M. Loss, and J. P. Solovej, Stability of matter in magnetic fields, *Phys. Rev. Lett.* **75**:985–989 (1995).
11. E. H. Lieb, J. P. Solovej, and J. Yngvason, Asymptotics of heavy atoms in high magnetic fields: I. Lowest Landau band region, *Comm. Pure Appl. Math.* **47**:513–591 (1994).
12. E. H. Lieb, J. P. Solovej, and J. Yngvason, Asymptotics of heavy atoms in high magnetic fields: II. Semiclassical regions, *Comm. Math. Phys.* **161**:77–124 (1994).
13. E. H. Lieb, J. P. Solovej, and J. Yngvason, Ground states of large quantum dots in magnetic fields, *Phys. Rev. B* **51**:10646–10665 (1995).
14. E. H. Lieb and W. Thirring, Inequalities for moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, in *Studies in Mathematical Physics*, E. Lieb, B. Simon, and A. Wightman, eds. (Princeton University Press, 1975), pp. 269–330.
15. P. Li and S. T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156**:153–201 (1986).
16. M. Loss and H.-T. Yau, Stability of Coulomb systems with magnetic fields: III. Zero energy bound states of the Pauli operator, *Comm. Math. Phys.* **104**:283–290 (1986).
17. Z. Shen, On the moments of negative eigenvalues for the Pauli operator, *J. Differential Equations* **149**:292–327 (1998) and **151**:420–455 (1999).

18. B. Simon, *Functional Integration and Quantum Physics* (Academic Press, New York, 1979).
19. A. Sobolev, On the Lieb–Thirring estimates for the Pauli operator, *Duke Math. J.* **82**:607–635 (1996).
20. A. Sobolev, Lieb–Thirring inequalities for the Pauli operator in three dimensions, *IMA Vol. Math. Appl.* **95**:155–188 (1997).
21. A. Sobolev, Quasiclassical asymptotics for the Pauli operator, *Comm. Math. Phys.* **194**:109–134 (1998).